A (condensed) primer on PAC-Bayesian Learning
followed by
A walkthrough of advanced PAC-Bayes results

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Foundational AI Seminar Series
June 16, 2020

Greetings!

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- Principal research fellow ($\sim$ associate professor) at UCL CS and AI,
- Tenured research scientist at Inria (Lille - Nord Europe),
- Scientific director of the Inria London joint lab with UCL CS and AI,
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Most recent research: coupling machine learning and sleep deprivation.

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- Cover all of our ICML 2019 tutorial!
  See `https://bguedj.github.io/icml2019/index.html`
- Cover our NIPS 2017 workshop "(Almost) 50 Shades of Bayesian Learning: PAC-Bayesian trends and insights"
  See `https://bguedj.github.io/nips2017/`

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MSc interns, PhD students, postdocs, visiting researchers

Part I

A Primer on PAC-Bayesian Learning

ICML 2019 tutorial

John

https://bguedj.github.io/icml2019/index.html

Survey in the Journal of the French Mathematical Society: *Guedj (2019)*

Learning is to be able to generalise

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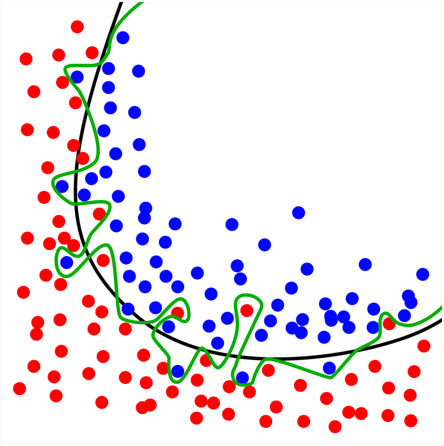
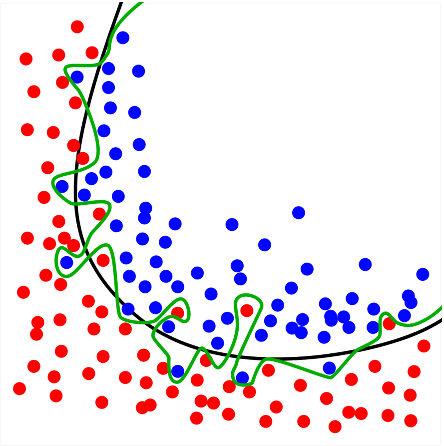


[Figure from Wikipedia]

Learning is to be able to generalise



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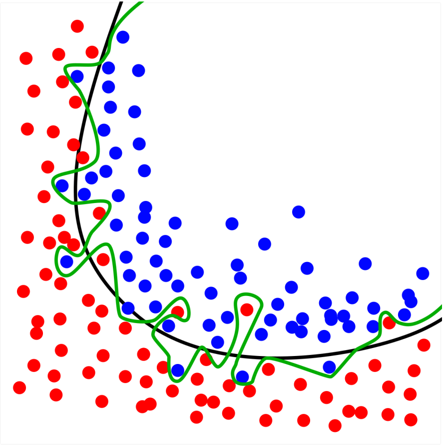

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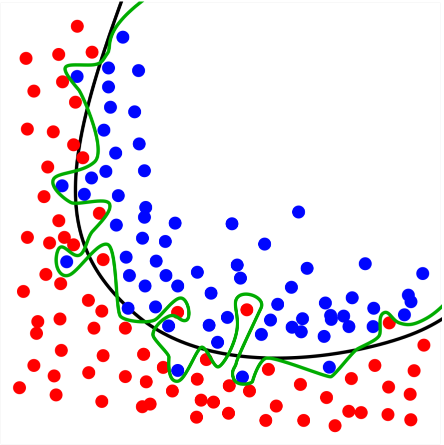

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Generalisation is the ability to 'perform' well on unseen data.

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- Hence high confidence: $\mathbb{P}^m[\text{approximately correct}] \geqslant 1 - \delta$

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Learning algorithm $\mathsf{A} : \mathcal{Z}^m \to \mathcal{H}$

- $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$
  $\mathcal{X}$ = set of inputs
  $\mathcal{Y}$ = set of outputs (e.g. labels)

- $\mathcal{H}$ = hypothesis class
  = set of predictors
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- Data-generating distribution $\mathbb{P}$ over $\mathcal{Z}$
- Learner doesn't know $\mathbb{P}$, only sees the training set
- Examples are *i.i.d.*: $S_m \sim \mathbb{P}^m$

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Actually these two goals interact with each other!

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Examples:

- $\ell(h(X), Y) = \mathbf{1}[h(X) \neq Y]$: 0-1 loss (classification)
- $\ell(h(X), Y) = (Y - h(X))^2$: square loss (regression)
- $\ell(h(X), Y) = (1 - Yh(X))_+$: hinge loss
- $\ell(h(X), 1) = -\log(h(X))$: log loss (density estimation)

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Flavours:

- distribution-free
- algorithm-free
- distribution-dependent
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- explain why specific learning algorithms actually work
- and even lead to designing new algorithm which scale to more complex settings

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These approaches are suited to analyse the performance of individual functions, and take some account of correlations.
$\longrightarrow$ Extension: PAC-Bayes allows to consider *distributions* over hypotheses.

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The risk measures $R_{\mathrm{in}}(h)$ and $R_{\mathrm{out}}(h)$ are extended by averaging:

$$R_{\mathrm{in}}(Q) \equiv \int_{\mathcal{H}} R_{\mathrm{in}}(h)\, dQ(h) \qquad R_{\mathrm{out}}(Q) \equiv \int_{\mathcal{H}} R_{\mathrm{out}}(h)\, dQ(h)$$

$\mathrm{KL}(Q \| P) = \mathop{\mathbf{E}}_{h \sim Q} \ln \frac{Q(h)}{P(h)}$ is the Kullback-Leibler divergence.

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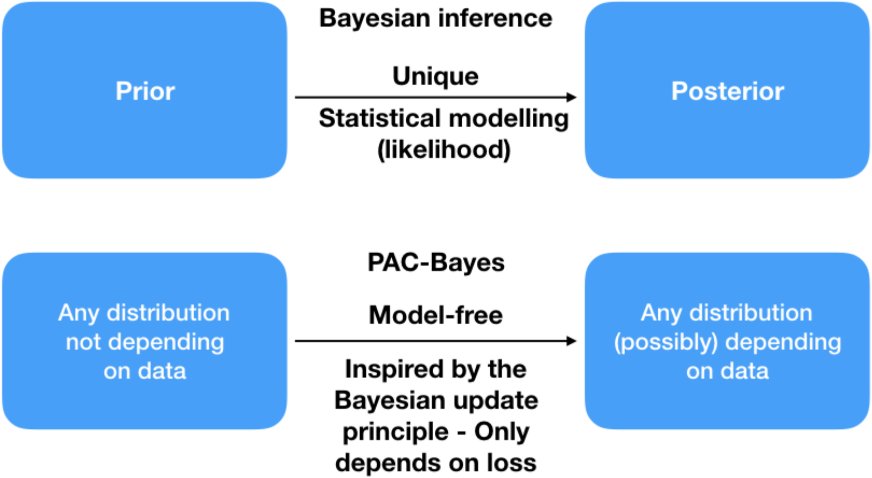

"Prior": exploration mechanism of $\mathcal{H}$

"Posterior" is the twisted prior after confronting with data

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- Data distribution
  - PAC-Bayes: bounds hold for any distribution
  - Bayes: randomness lies in the noise model generating the output

A classical PAC-Bayesian bound

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Pre-history: PAC analysis of Bayesian estimators

Shawe-Taylor and Williamson (1997); Shawe-Taylor et al. (1998)

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Shawe-Taylor and Williamson (1997); Shawe-Taylor et al. (1998)

Birth: PAC-Bayesian bound

McAllester (1998, 1999)

McAllester Bound

For any prior P, any $\delta \in (0,1]$, we have

$$\mathbb{P}^m\left(\forall\, Q \text{ on } \mathcal{H}\colon\ R_{\text{out}}(Q) \leqslant R_{\text{in}}(Q) + \sqrt{\frac{\mathrm{KL}(Q\|P) + \ln\frac{2\sqrt{m}}{\delta}}{2m}}\right) \geqslant 1-\delta\,,$$

A flexible framework

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Since 1997, PAC-Bayes has been successfully used in many machine learning settings (this list is by no means exhaustive).

Statistical learning theory *Shawe-Taylor and Williamson (1997); McAllester (1998, 1999, 2003a,b); Seeger (2002, 2003); Maurer (2004); Catoni (2004, 2007); Audibert and Bousquet (2007); Thiemann et al. (2017); Guedj (2019); Mhammedi et al. (2019); Guedj and Pujol (2019); Haddouche et al. (2020)*

SVMs & linear classifiers *Langford and Shawe-Taylor (2002); McAllester (2003a); Germain et al. (2009a)*

Supervised learning algorithms reinterpreted as bound minimizers *Ambroladze et al. (2007); Shawe-Taylor and Hardoon (2009); Germain et al. (2009b)*

High-dimensional regression *Alquier and Lounici (2011); Alquier and Biau (2013); Guedj and Alquier (2013); Li et al. (2013); Guedj and Robbiano (2018)*

Classification *Langford and Shawe-Taylor (2002); Catoni (2004, 2007); Lacasse et al. (2007); Parrado-Hernández et al. (2012)*

A flexible framework

- Transductive learning, domain adaptation *Derbeko et al. (2004); Bégin et al. (2014); Germain et al. (2016); Nozawa et al. (2020)*
- Non-iid or heavy-tailed data *Lever et al. (2010); Seldin et al. (2011, 2012); Alquier and Guedj (2018); Holland (2019)*
- Density estimation *Seldin and Tishby (2010); Higgs and Shawe-Taylor (2010)*
- Reinforcement learning *Fard and Pineau (2010); Fard et al. (2011); Seldin et al. (2011, 2012); Ghavamzadeh et al. (2015)*
- Sequential learning *Gerchinovitz (2011); Li et al. (2018)*
- Algorithmic stability, differential privacy *London et al. (2014); London (2017); Dziugaite and Roy (2018a,b); Rivasplata et al. (2018)*
- Deep neural networks *Dziugaite and Roy (2017); Neyshabur et al. (2017); Zhou et al. (2019); Letarte et al. (2019); Biggs and Guedj (2020)*
- ...

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SVMs, KL-regularized Adaboost, exponential weights are all minimisers of PAC-Bayes bounds.

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(i) For any probability P on $(A, \mathcal{A})$ and any measurable function $\phi : A \to \mathbb{R}$ such that $\int (\exp \circ \phi) \mathrm{d}P < \infty$,

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(ii) If ϕ is upper-bounded on the support of P, the supremum is reached for the Gibbs distribution G given by

$$\frac{\mathrm{d}G}{\mathrm{d}P}(a) = \frac{\exp \circ \phi(a)}{\int (\exp \circ \phi) \mathrm{d}P}, \quad a \in A.$$

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Let $\lambda > 0$ and take $\phi = -\lambda R_{\mathrm{in}}$,

$$Q_\lambda \propto \exp \left(-\lambda R_{\mathrm{in}} \right) P = \operatorname*{arg\,inf}_{Q \ll P} \left\{ R_{\mathrm{in}}(Q) + \frac{\mathrm{KL}(Q, P)}{\lambda} \right\}.$$

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What is coming next

- A small sample of what PAC-Bayes can offer!

Part II

A (gentle) walkthrough of state-of-the-art PAC-Bayes

- Guedj and Robbiano (2018). PAC-Bayesian high dimensional bipartite ranking, Journal of Statistical Planning and Inference.
- Alquier and Guedj (2018). Simpler PAC-Bayesian bounds for hostile data, Machine Learning.
- Letarte, Germain, Guedj and Laviolette (2019). Dichotomize and generalize: PAC-Bayesian binary activated deep neural networks, NeurIPS 2019.
- Nozawa, Germain and Guedj (2020). PAC-Bayesian contrastive unsupervised representation learning, UAI.





Bipartite ranking

- $(\mathbf{X}, Y) \in \mathbb{R}^d \times \{\pm 1\}$, iid replications $\{(\mathbf{X}_i, Y_i)\}_{i=1}^m$
- $\eta(\cdot) = \mathbb{P}\{Y = 1 | \mathbf{X} = \cdot\}$

Goal: design an order relationship on $\mathbb{R}^d$ which is consistent with the order on $\{\pm 1\}$. Scoring function $s \colon \mathbb{R}^d \to \mathbb{R}$

$$\forall (\mathbf{x}, \mathbf{x}') \in \mathbb{R}^d \times \mathbb{R}^d, \qquad \mathbf{x} \preceq_s \mathbf{x}' \Leftrightarrow s(\mathbf{x}) \leqslant s(\mathbf{x}').$$

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Idea: build s such that

$$\forall (\mathbf{x}, \mathbf{x}') \in \mathbb{R}^d \times \mathbb{R}^d, \quad s(\mathbf{x}) \leqslant s(\mathbf{x}') \Leftrightarrow \eta(\mathbf{x}) \leqslant \eta(\mathbf{x}').$$

PAC-Bayes ranking

Ranking risk of a scoring function s and empirical counterpart

$$L(s) = \mathbb{P}\left[(s(\mathbf{X}) - s(\mathbf{X}'))(Y - Y') < 0\right].$$

$$L_m(s) = \frac{1}{m(m-1)} \sum_{i \neq j} \mathbf{1}_{\{(s(\mathbf{X}_i) - s(\mathbf{X}_j))(Y_i - Y_j) < 0\}}.$$

Dictionary of deterministic functions $\mathbb{D} = \{\phi_1, \ldots, \phi_M\}$,

$$\mathcal{S}_\Theta = \left\{ s_\theta \colon \mathbf{x} \mapsto \sum_{j=1}^{d} \sum_{k=1}^{M} \theta_{jk} \phi_k(x_j) = \langle \theta, \mathbb{D}(\mathbf{x}) \rangle, \quad \theta \in \mathbb{R}^{dM} \right\}.$$

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Gibbs measure $Q_\lambda(\mathrm{d}\theta) \propto \exp[-\lambda L_n(s_\theta)] P(\mathrm{d}\theta)$, $\lambda > 0$.

PAC-Bayes predictor

$$\widehat{s} = s_{\widehat{\theta}} : \mathbf{x} \mapsto \sum_{j=1}^{d} \sum_{k=1}^{M} \widehat{\theta}_{jk} \phi_k(x_j) = \langle \widehat{\theta}, \mathbb{D}(\mathbf{x}) \rangle, \quad \widehat{\theta} \sim Q_\lambda.$$

MCMC implementation (Metropolised Carlin and Chib)

Oracle generalisation bounds

For any distribution of $(\mathbf{X}, Y)$, any prior P, any $\delta \in (0, 1)$,

$$\mathbb{P}\left[L(\widehat{s}) - L(\eta) \leqslant \inf_{Q \ll P} \left\{\int L(s) Q(\mathrm{d}s) - L(\eta) \right.\right.$$
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Under a margin condition on η, we proved the first minimax optimal rates for high dimensional bipartite ranking.

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Csiszár f-divergence: let f be a convex function with $f(1) = 0$,

$$D_f(Q, P) = \int f\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right) \mathrm{d}P$$

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The KL is given by the special case $\mathrm{KL}(Q\|P) = D_{x \log(x)}(Q, P)$.

Power function: $\phi_p \colon x \mapsto x^p$.

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- the moment $\mathcal{M}_q$ (which depends on the distribution of the data)
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For $p = q = 2$, w.p. $\geqslant 1-\delta$, $R_{\mathrm{out}}(Q) \leqslant R_{\mathrm{in}}(Q) + \sqrt{\frac{\mathcal{V}}{m\delta} \int \left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right)^2 \mathrm{d}P}$.

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\textbf{Change of measure} \quad & = \int \Delta \frac{\mathrm{d}Q}{\mathrm{d}P} \mathrm{d}P \\
\textbf{Hölder} \quad & \leqslant \left(\int \Delta^q \mathrm{d}P \right)^{\frac{1}{q}} \left(\int \left(\frac{\mathrm{d}Q}{\mathrm{d}P} \right)^p \mathrm{d}P \right)^{\frac{1}{p}} \\
\textbf{Markov} \quad & \underset{1-\delta}{\leqslant} \left(\frac{\mathbb{E} \int \Delta^q \mathrm{d}P}{\delta} \right)^{\frac{1}{q}} \left(\int \left(\frac{\mathrm{d}Q}{\mathrm{d}P} \right)^p \mathrm{d}P \right)^{\frac{1}{p}} \\
& = \left(\frac{\mathcal{M}_q}{\delta} \right)^{\frac{1}{q}} \left(D_{\phi_p - 1}(Q, P) + 1 \right)^{\frac{1}{p}} .
\end{aligned}
$$

Standard Neural Networks

Classification setting:

- $\mathbf{x} \in \mathbb{R}^{d_0}$
- $y \in \{-1, 1\}$

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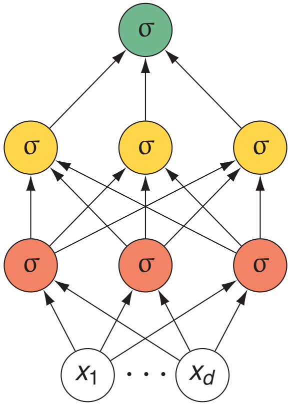
- *L fully connected* layers
- d_k denotes the number of neurons of the k^{th} layer
- $\sigma : \mathbb{R} \to \mathbb{R}$ is the *activation function*

Parameters:

- $\mathbf{W}_k \in \mathbb{R}^{d_k \times d_{k-1}}$ denotes the weight matrices, $D = \sum_{k=1}^{L} d_{k-1} d_k$.
- $\theta = \mathrm{vec}\left(\{\mathbf{W}_k\}_{k=1}^{L}\right) \in \mathbb{R}^D$



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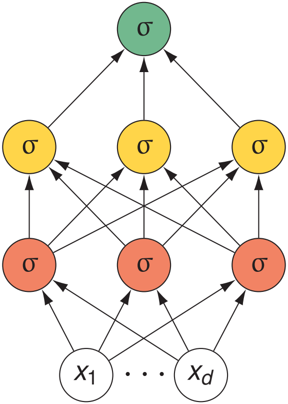


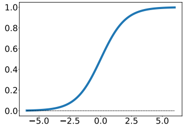
Prediction

$$f_\theta(\mathbf{x}) = \sigma\Big(\mathbf{w}_L \sigma\Big(\mathbf{W}_{L-1} \sigma\Big(\ldots \sigma\Big(\mathbf{W}_1 \mathbf{x}\Big)\Big)\Big)\Big)\,.$$

PAC-Bayesian bounds for Stochastic NN

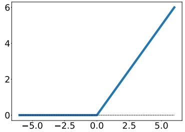
Langford and Caruana (2001)

- Shallow networks ($L = 2$)
- Sigmoid activation functions



Dziugaite and Roy (2017)

- Deep networks ($L > 2$)
- ReLU activation functions



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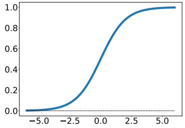

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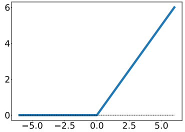


Idea: Bound the expected loss of the network under a Gaussian perturbation of the weights

Empirical loss: $\underset{\theta' \sim \mathcal{N}(\theta,\Sigma)}{\mathbf{E}} R_{\mathrm{in}}(f_{\theta'})$ $\longrightarrow$ estimated by sampling

Complexity term: $\mathrm{KL}(\mathcal{N}(\theta, \Sigma) \| \mathcal{N}(\theta_0, \Sigma_0))$ $\longrightarrow$ closed form

Binary Activated Neural Networks

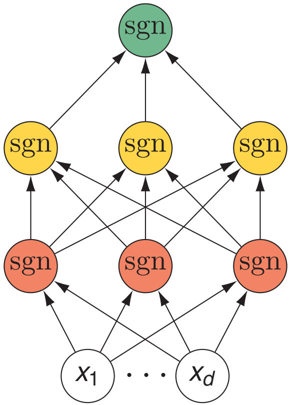
- $\mathbf{x} \in \mathbb{R}^{d_0}$
- $y \in \{-1, 1\}$

Architecture:

- *L fully connected* layers
- d_k denotes the number of neurons of the k^{th} layer
- $\mathrm{sgn}(a) = 1$ if $a > 0$ and $\mathrm{sgn}(a) = -1$ otherwise

Parameters:

- $\mathbf{W}_k \in \mathbb{R}^{d_k \times d_{k-1}}$ denotes the weight matrices.
- $\theta = \mathrm{vec}\big(\{\mathbf{W}_k\}_{k=1}^{L}\big) \in \mathbb{R}^D$



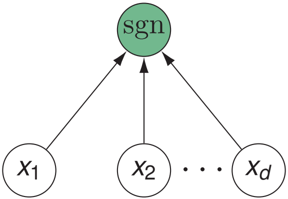
Prediction

$$f_\theta(\mathbf{x}) = \mathrm{sgn}\Big(\mathbf{w}_L \mathrm{sgn}\Big(\mathbf{W}_{L-1} \mathrm{sgn}\Big(\ldots \mathrm{sgn}\Big(\mathbf{W}_1 \mathbf{x}\Big)\Big)\Big)\Big),$$

One Layer (linear predictor)

Germain et al. (2009a)

$$f_{\mathbf{w}}(\mathbf{x}) \stackrel{\text{def}}{=} \mathrm{sgn}(\mathbf{w} \cdot \mathbf{x}), \text{ with } \mathbf{w} \in \mathbb{R}^{d_0}.$$



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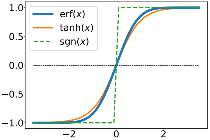

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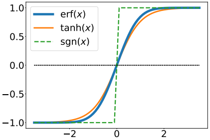

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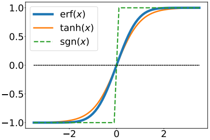

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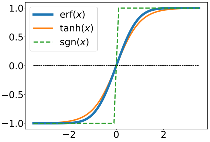

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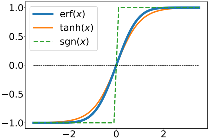

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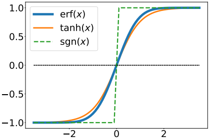
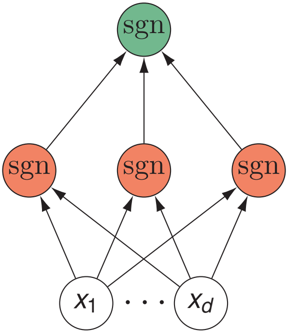


Bound minimisation — under the linear loss $\ell(y, y') := \frac{1}{2}(1 - yy')$

$$C m R_{\text{in}}(F_{\mathbf{w}}) + \mathrm{KL}(Q_{\mathbf{w}} \| P_{\mathbf{w}_0}) = C \, \frac{1}{2} \sum_{i=1}^{m} \operatorname{erf}\left(-y_i \, \frac{\mathbf{w} \cdot \mathbf{x}_i}{\sqrt{d}\|\mathbf{x}_i\|}\right) + \frac{1}{2}\|\mathbf{w} - \mathbf{w}_0\|^2 .$$

Two Layers (shallow network)



Two Layers (shallow network)

Posterior $Q_\theta = \mathcal{N}(\theta, I_D)$, over the family of all networks $\mathcal{F}_D = \{f_{\tilde{\theta}} \,|\, \tilde{\theta} \in \mathbb{R}^D\}$, where

$$f_\theta(\mathbf{x}) = \operatorname{sgn}\big(\mathbf{w}_2 \cdot \operatorname{sgn}(\mathbf{W}_1 \mathbf{x})\big) \, .$$

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$$F_\theta(\mathbf{x}) = \mathop{\mathbf{E}}_{\tilde\theta \sim Q_\theta} f_{\tilde\theta}(\mathbf{x})$$

Two Layers (shallow network)

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$$\begin{aligned}
F_\theta(\mathbf{x}) &= \mathop{\mathbf{E}}_{\tilde\theta\sim Q_\theta} f_{\tilde\theta}(\mathbf{x}) \\
&= \int_{\mathbb{R}^{d_1\times d_0}} Q_1(\mathbf{V}_1) \int_{\mathbb{R}^{d_1}} Q_2(\mathbf{v}_2)\operatorname{sgn}(\mathbf{v}_2 \cdot \operatorname{sgn}(\mathbf{V}_1\mathbf{x}))\, d\mathbf{v}_2\, d\mathbf{V}_1 \\
&= \int_{\mathbb{R}^{d_1\times d_0}} Q_1(\mathbf{V}_1)\operatorname{erf}\left(\frac{\mathbf{w}_2\cdot\operatorname{sgn}(\mathbf{V}_1\mathbf{x})}{\sqrt{2}\|\operatorname{sgn}(\mathbf{V}_1\mathbf{x})\|}\right) d\mathbf{V}_1 \\
&= \sum_{\mathbf{s}\in\{-1,1\}^{d_1}} \operatorname{erf}\left(\frac{\mathbf{w}_2\cdot\mathbf{s}}{\sqrt{2d_1}}\right) \int_{\mathbb{R}^{d_1\times d_0}} \mathbb{1}[\mathbf{s} = \operatorname{sgn}(\mathbf{V}_1\mathbf{x})]\, Q_1(\mathbf{V}_1)\, d\mathbf{V}_1 \\
&= \sum_{\mathbf{s}\in\{-1,1\}^{d_1}} \underbrace{\operatorname{erf}\left(\frac{\mathbf{w}_2\cdot\mathbf{s}}{\sqrt{2d_1}}\right)}_{F_{\mathbf{w}_2}(\mathbf{s})} \underbrace{\prod_{i=1}^{d_1}\left[\frac{1}{2} + \frac{s_i}{2}\operatorname{erf}\left(\frac{\mathbf{w}_1^i\cdot\mathbf{x}}{\sqrt{2}\,\|\mathbf{x}\|}\right)\right]}_{\Pr(\mathbf{s}|\mathbf{x},\mathbf{W}_1)}.
\end{aligned}$$

Stochastic Approximation

$$F_\theta(\mathbf{x}) = \sum_{\mathbf{s}\in\{-1,1\}^{d_1}} F_{\mathbf{w}_2}(\mathbf{s}) \Pr(\mathbf{s}|\mathbf{x}, \mathbf{W}_1)$$

Monte Carlo sampling

We generate T random binary vectors $\{\mathbf{s}^t\}_{t=1}^T$ according to $\Pr(\mathbf{s}|\mathbf{x}, \mathbf{W}_1)$

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Prediction.

$$F_{\theta}(\mathbf{x}) \approx \frac{1}{T}\sum_{t=1}^{T} F_{\mathbf{w}_2}(\mathbf{s}^t) \,.$$

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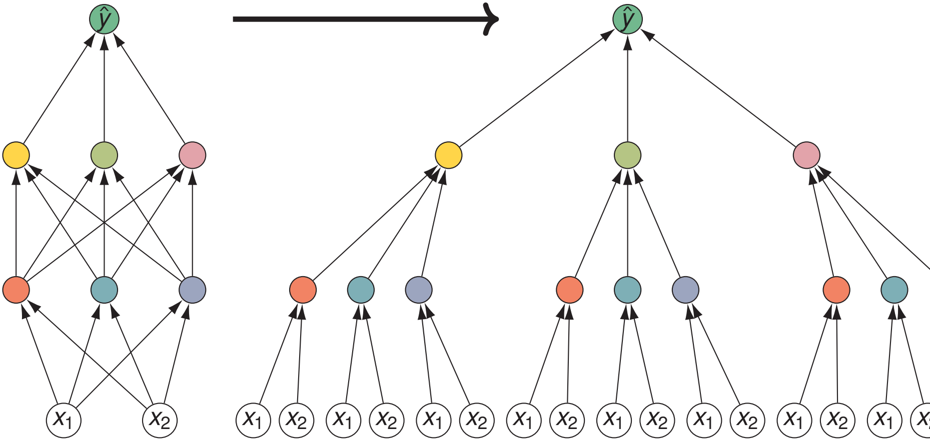
Derivatives.

$$\frac{\partial}{\partial \mathbf{w}_1^k} F_\theta(\mathbf{x}) \approx \frac{\mathbf{x}}{2^{\frac{3}{2}}\|\mathbf{x}\|} \mathrm{erf}'\left(\frac{\mathbf{w}_1^k \cdot \mathbf{x}}{\sqrt{2}\,\|\mathbf{x}\|}\right) \frac{1}{T}\sum_{t=1}^{T} \frac{s_k^t}{\Pr(s_k^t|\mathbf{x}, \mathbf{w}_1^k)} F_{\mathbf{w}_2}(\mathbf{s}^t) .$$

More Layers (deep)



$$F_1^{(j)}(\mathbf{x}) = \operatorname{erf}\left(\frac{\mathbf{w}_1^j \cdot \mathbf{x}}{\sqrt{2}\|\mathbf{x}\|}\right), \qquad F_{k+1}^{(j)}(\mathbf{x}) = \sum_{\mathbf{s}\in\{-1,1\}^{d_k}} \operatorname{erf}\left(\frac{\mathbf{w}_{k+1}^j \cdot \mathbf{s}}{\sqrt{2d_k}}\right) \prod_{i=1}^{d_k} \left(\frac{1}{2} + \frac{1}{2} s_i \times F_k^{(i)}(\mathbf{x})\right)$$

Generalisation bound

Let G_θ denote the predictor with posterior mean as parameters.
With probability at least $1-\delta$, for any $\theta \in \mathbb{R}^D$

$$R_{\mathrm{out}}(G_\theta) \leqslant$$
$$\inf_{C>0}\left\{\frac{1}{1-e^{-C}}\left(1-\exp\left(-CR_{\mathrm{in}}(G_\theta)-\frac{\mathrm{KL}(\theta,\theta_0)+\log\frac{2\sqrt{m}}{\delta}}{m}\right)\right)\right\}.$$

Numerical results

Model name	Cost function	Train split	Valid split	Model selection	Prior
MLP–tanh	linear loss, L2 regularized	80%	20%	valid linear loss	-
PBGNet$_\ell$	linear loss, L2 regularized	80%	20%	valid linear loss	random init
PBGNet	**PAC-Bayes bound**	**100 %**	**-**	**PAC-Bayes bound**	**random init**
PBGNet$_{\text{pre}}$					
– pretrain	linear loss (20 epochs)	50%	-	-	random init
– final	PAC-Bayes bound	50%	-	PAC-Bayes bound	pretrain

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Dataset	MLP–tanh		PBGNet$_\ell$		PBGNet			PBGNet$_{\text{pre}}$		
	E_S	E_T	E_S	E_T	E_S	E_T	Bound	E_S	E_T	Bound
ads	0.021	0.037	0.018	**0.032**	0.024	0.038	0.283	0.034	0.033	0.058
adult	0.128	0.149	0.136	**0.148**	0.158	0.154	0.227	0.153	0.151	0.165
mnist17	0.003	**0.004**	0.008	0.005	0.007	0.009	0.067	0.003	0.005	0.009
mnist49	0.002	**0.013**	0.003	0.018	0.034	0.039	0.153	0.018	0.021	0.030
mnist56	0.002	0.009	0.002	0.009	0.022	0.026	0.103	0.008	**0.008**	0.017
mnistLH	0.004	**0.017**	0.005	0.019	0.071	0.073	0.186	0.026	0.026	0.033

Contrastive unsupervised representation learning (aka CURL)

SOTA technique to learn representations (as a set of features) from unlabelled data (e.g., word2vec, image classification). Contrastive loss differentiates inputs by similarity.

Arora et al. (2019): first theoretical results on CURL, using Rademacher complexity. In a nutshell, for any predictor f and $\widehat{f}$ an ERM, w.p. $\geqslant 1-\delta$,

$$\mathrm{Loss}_{\mathrm{sup}}(\widehat{f}) \leqslant C_1 \mathrm{Loss}_{\mathrm{uns}}(f) + C_2 \left(\frac{\mathrm{Rad}}{m} + \sqrt{\frac{\log(1/\delta)}{m}} \right).$$

We proposed a PAC-Bayes version which improves on their results by removing the iid assumption and by deriving a SOTA learning algorithm. For any prior P, any posterior Q, any $\lambda > 0$, w.p. $\geqslant 1-\delta$

$$\mathrm{Loss}_{\mathrm{sup}}(Q) \leqslant C \left(\frac{1-\exp\left(-\lambda \widehat{\mathrm{Loss}}_{\mathrm{uns}}(Q) - \frac{\mathrm{KL}(Q,P)+\log(1/\delta)}{m}\right)}{1-\exp(-\lambda)} \right).$$

Thanks!

What this talk could have been about...

- ☐ Tighter PAC-Bayes bounds (Mhammedi et al., 2019)
- ☐ PAC-Bayes and robust learning (Guedj and Pujol, 2019; Haddouche et al., 2020)
- ☐ PAC-Bayesian online clustering (Li et al., 2018)
- ■ Online *k*-means clustering (Cohen-Addad et al., 2019)
- ■ Sequential learning of principal curves (Guedj and Li, 2018)
- ■ Stability and generalisation (Celisse and Guedj, 2016)
- ■ Decentralised learning with aggregation (Klein et al., 2019)
- ■ Image denoising (Guedj and Rengot, 2020)
- ■ Matrix factorisation (Alquier and Guedj, 2017; Chrétien and Guedj, 2020)
- ■ Preventing model overfitting (Zhang et al., 2019)
- ■ / ☐ a few others... (very) soon on arXiv

☐ = PAC-Bayes

■ = "Wait, I can talk about other stuff too!"

This talk:

`https://bguedj.github.io/talks/2020-06-16-seminar-faicdt`

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